

VARIATION OF LOCAL SYSTEMS AND PARABOLIC COHOMOLOGY

BY

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ABSTRACT

Given a family of local systems on a punctured Riemann sphere, with moving singularities, its first parabolic cohomology is a local system on the base space. We study this situation from different points of view. For instance, we derive universal formulas for the monodromy of the resulting local system. We use a particular example of our construction to prove that the simple groups $\mathrm{PSL}_2(p^2)$ admit regular realizations over the field $\mathbb{Q}(t)$ for primes $p \not\equiv 1, 4, 16 \pmod{21}$.

Introduction

Local systems on the punctured Riemann sphere arise in various branches of mathematics and have been intensively studied (see Section 1 for the definition of a local system and [5], [16]). One way to produce such local systems with interesting properties is the following. Suppose we are given a family \mathcal{V}_s of local systems on the punctured sphere with moving singularities, parameterized by some base space S . More precisely, let $D \subset \mathbb{P}^1 \times S$ be a smooth relative divisor and let \mathcal{V} be a local system on $U := \mathbb{P}^1 \times S - D$. Let $\bar{\pi}: \mathbb{P}^1 \times S \rightarrow S$ denote the

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second projection and $j: U \hookrightarrow \mathbb{P}^1 \times S$ the canonical inclusion. Then the first higher direct image sheaf

$$\mathcal{W} := R^1 \pi_* (j_* \mathcal{V})$$

is a local system on S , whose stalk at a point $s \in S$ is the first parabolic cohomology group of the local system \mathcal{V}_s , the restriction of \mathcal{V} to the fiber $U_s := \pi^{-1}(s)$. Choose a base point $s_0 \in S$ and set $\mathcal{V}_0 := \mathcal{V}_{s_0}$. We call \mathcal{V} a **variation** of the local system \mathcal{V}_0 over the base S and \mathcal{W} the first parabolic cohomology of the variation \mathcal{V} .

A special case of this construction is the **middle convolution** studied by N. Katz in [16] and by Dettweiler and Reiter in [11]. Starting with some local system \mathcal{V}_0 on the punctured Riemann sphere U_0 , one constructs a variation of local systems over the base U_0 by twisting \mathcal{V}_0 with a one-dimensional system with two singularities, one of which is moving over all points of U_0 . The parabolic cohomology of this variation gives rise to a local system on U_0 , called the **middle convolution** of \mathcal{V}_0 . In [16], Katz proves that all rigid local systems can be constructed from one-dimensional systems by successive application of middle convolution and ‘scaling’.

A local system on the punctured Riemann sphere corresponds to an r -tuple of invertible matrices $\mathbf{g} = (g_1, \dots, g_r) \in \mathrm{GL}_n(\mathbb{C})^r$. Völklein [26] and, independently, Dettweiler and Reiter [10] have defined an operation $\mathbf{g} \mapsto \tilde{\mathbf{g}}$ on tuples of invertible matrices over any field K corresponding to the middle convolution (in [26], it is called the **braid companion functor**). The definition of this operation needs only simple linear algebra. Therefore, the tuple $\tilde{\mathbf{g}}$ can be easily computed, whereas in the original work of Katz the matrices \tilde{g}_i are computed only up to conjugation in GL_m . This construction has had many applications to the Regular Inverse Galois Problem; see, e.g., [26] and [10]

The goal of the present paper is to study the parabolic cohomology of an arbitrary variation of local systems (see the beginning of this introduction), both from an analytic and from an arithmetic point of view.

In the first part of our paper, we treat the analytic aspect, i.e. we use singular cohomology. Given a variation \mathcal{V} of local systems over a base S , we present an effective method to compute the monodromy representation of $\pi_1(S)$ on the parabolic cohomology of \mathcal{V} . The result depends on the tuple of matrices corresponding to the fibers of the variation \mathcal{V} , and on the map from $\pi_1(S)$ to the Hurwitz braid group which describes how the singularities of these fibres move around on \mathbb{P}^1 . Formally, our method is a straightforward extension of Völklein’s braid companion functor. Apart from the greater generality, the main difference

to Völklein's approach is that we provide a cohomological interpretation of our computation. Such an interpretation has the advantage that it makes it very easy, using comparison theorems, to translate results from one world into another. For instance, one can use topological methods to compute the geometric monodromy of an étale local system.

To illustrate this, we give an example which is concerned with the Regular Inverse Galois Problem. We use a Hurwitz family which arises from the Klein group of order 168 to construct a variation of local systems over the underlying Hurwitz curve. This Hurwitz curve turns out to be the projective line over \mathbb{Q} with 17 points removed. The parabolic cohomology of this variation then gives rise to a family of two-dimensional Galois representations of the algebraic fundamental group of this punctured projective line. Using reduction modulo \mathfrak{p} , we obtain the following result:

THEOREM: *The simple group $\mathrm{PSL}_2(p^2)$ admits a regular realization as Galois group over $\mathbb{Q}(t)$, for all primes $p \not\equiv 1, 4, 16 \pmod{21}$.*

The only other cases where the group $\mathrm{PSL}_2(p^2)$ is known to admit a regular realization over $\mathbb{Q}(t)$ are for $p \not\equiv \pm 1 \pmod{5}$, by a result of Feit [12], and for $p \not\equiv \pm 1 \pmod{24}$, by a result of Shiina [22], [23] (see also [18], [7] and [27]).

If $q < n$, then one knows that the group $\mathrm{PSP}_{2n}(q)$ occurs regularly over $\mathbb{Q}(t)$; see [24] and [10]. Similar bounds exist also for other classical groups. On the other hand, experience shows that it is much harder to realize classical groups of small Lie rank. The realizations of $\mathrm{PSL}_2(p^2)$ in the above theorem all come from one particular variation of local systems. It is very likely that, by choosing different variations, one can realize many more series of classical groups of small rank. We have chosen one particular example leading to the above theorem, because the case of rank one seems to us the hardest case.

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1. Parabolic cohomology

We study the first parabolic cohomology of a local system on the punctured sphere. In particular, we show that it is isomorphic to a certain module $W_{\mathbf{g}}$, defined in [26].

1.1. Let X be a connected and locally contractible topological space. Let R be a commutative ring with unit. A **local system** of R -modules on X is a locally constant sheaf \mathcal{V} on X whose stalks are free R -modules of finite rank. We denote by \mathcal{V}_x the stalk of \mathcal{V} at a point $x \in X$. If $f: Y \rightarrow X$ is a continuous map, then \mathcal{V}_f denotes the group of global sections of the sheaf $f^*\mathcal{V}$. Note that if Y is simply connected, then the natural morphism

$$\mathcal{V}_f \longrightarrow \mathcal{V}_{f(y)}$$

is an isomorphism, for all $y \in Y$. Therefore, a path $\alpha: [0, 1] \rightarrow X$ gives rise to an isomorphism

$$\mathcal{V}_{\alpha(0)} \xrightarrow{\sim} \mathcal{V}_{\alpha(1)},$$

obtained as the composition of the isomorphisms $\mathcal{V}_{\alpha(0)} \cong \mathcal{V}_\alpha$ and $\mathcal{V}_\alpha \cong \mathcal{V}_{\alpha(1)}$. The image of $v \in \mathcal{V}_{\alpha(0)}$ under the above isomorphism is denoted by v^α . It only depends on the homotopy class of α .

Let us fix a base point $x_0 \in X$ and set $V := \mathcal{V}_{x_0}$. We let elements of $\text{GL}(V)$ act on V from the *right*. Then the map

$$\rho: \pi_1(X, x_0) \longrightarrow \text{GL}(V),$$

defined by $v \cdot \rho(\alpha) := v^\alpha$, is a group homomorphism, i.e. a representation of $\pi_1(X, x_0)$. It is a standard fact that the functor $\mathcal{V} \mapsto V := \mathcal{V}_{x_0}$ is an equivalence of categories between local systems on X and representations of $\pi_1(X, x_0)$.

1.2. Let X be a compact (topological) surface of genus 0 and $D \subset X$ a subset of cardinality r . We set $U := X - D$. There exists a homeomorphism $\kappa: X \xrightarrow{\sim} \mathbb{P}_\mathbb{C}^1$ between X and the Riemann sphere which maps the set D to the real line $\mathbb{R} \subset \mathbb{P}_\mathbb{C}^1$. Such a homeomorphism is called a **marking** of (X, D) .

Let us, for the moment, identify X with $\mathbb{P}_\mathbb{C}^1$ using the marking κ . Write $D = \{x_1, \dots, x_r\}$ with $x_1 < x_2 < \dots < x_r$ and choose a base point $x_0 \in U$ lying in the upper half plane. There is a standard presentation

$$(1) \quad \pi_1(U, x_0) = \langle \alpha_1, \dots, \alpha_r \mid \prod_i \alpha_i = 1 \rangle$$

of the fundamental group of U , depending only on κ . The generators α_i are generated by a simple closed loop which intersects the real line exactly twice, first the interval (x_{i-1}, x_i) , then the interval (x_i, x_{i+1}) .

Let \mathcal{V} be a local system of R -modules on U , corresponding to a representation $\rho: \pi_1(U, x_0) \rightarrow \text{GL}(V)$. For $i = 1, \dots, r$, set $g_i := \rho(\alpha_i) \in \text{GL}(V)$. Then we

have

$$\prod_{i=1}^r g_i = 1.$$

Conversely, given a free R -module V of finite rank and a tuple $\mathbf{g} = (g_1, \dots, g_r)$ of elements of $\text{GL}(V)$ satisfying the above relation, we obtain a local system \mathcal{V} which induces the tuple \mathbf{g} , as above.

1.3. We continue with the notation introduced in the previous subsection. Let $j: U \hookrightarrow X$ denote the inclusion. The **parabolic cohomology** of \mathcal{V} is defined as the sheaf cohomology of $j_*\mathcal{V}$, and is written as

$$H_p^n(U, \mathcal{V}) := H^n(X, j_*\mathcal{V}).$$

We have natural morphisms $H_c^n(U, \mathcal{V}) \rightarrow H_p^n(U, \mathcal{V})$ and $H_p^n(U, \mathcal{V}) \rightarrow H^n(U, \mathcal{V})$ (H_c denotes cohomology with compact support).

PROPOSITION 1.1:

- (i) *The group $H^n(U, \mathcal{V})$ is canonically isomorphic to the group cohomology $H^n(\pi_1(U, x_0), V)$. In particular, we have*

$$H^0(U, \mathcal{V}) \cong V^{(g_1, \dots, g_r)},$$

and $H^n(U, \mathcal{V}) = 0$ for $n > 1$.

- (ii) *The map $H_c^1(U, \mathcal{V}) \rightarrow H_p^1(U, \mathcal{V})$ is surjective and the map $H_p^1(U, \mathcal{V}) \rightarrow H^1(U, \mathcal{V})$ is injective. In other words, $H_p^1(U, \mathcal{V})$ is the image of the cohomology with compact support in $H^1(U, \mathcal{V})$.*

Proof: Part (i) follows from the Hochschild–Serre spectral sequence and the fact that the universal cover of U is contractible. For (ii), see e.g. [17], Lemma 5.3. ■

Let $\delta: \pi_1(U) \rightarrow V$ be a 1-cocycle, i.e. we have $\delta(\alpha\beta) = \delta(\alpha) \cdot \rho(\beta) + \delta(\beta)$. Set $v_i := \delta(\alpha_i)$. It is clear that the tuple (v_i) is subject to the relation

$$(2) \quad v_1 \cdot g_2 \cdots g_r + v_2 \cdot g_3 \cdots g_r + \cdots + v_r = 0.$$

Conversely, any tuple (v_i) satisfying (2) gives rise to a unique 1-cocycle δ . This cocycle is a coboundary if and only if there exists $v \in V$ such that $v_i = v \cdot (g_i - 1)$ for all i . By Proposition 1.1 there is a natural inclusion

$$H_p^1(U, \mathcal{V}) \hookrightarrow H^1(\pi_1(U), V).$$

We say that δ is a **parabolic** 1-cocycle if the class of δ in $H^1(\pi_1(U), V)$ lies in the image of $H_p^1(U, \mathcal{V})$.

LEMMA 1.2: *The 1-cocycle δ is parabolic if and only if v_i lies in the image of $g_i - 1$, for all i .*

Proof: Let $U_i \subset X$ be pairwise disjoint disks with center x_i , and set $U_i^* := U_i - \{x_i\}$. We have a long exact sequence

$$(3) \quad \cdots \rightarrow H_{x_i}^n(U_i, (j)\mathcal{V})|_{U_i} \rightarrow H^n(U_i, (j)\mathcal{V})|_{U_i} \rightarrow H^n(U_i^*, \mathcal{V}|_{U_i^*}) \rightarrow \cdots.$$

Given a class c in $H^n(U_i, (j)\mathcal{V})|_{U_i}$ we can find a smaller disk $U'_i \subset U_i$ with center x_i such that the restriction of c to U'_i vanishes (one way to see this is to use Čech cohomology). On the other hand, the cohomology groups $H_{x_i}^n(U_i, (j)\mathcal{V})|_{U_i}$ and $H^n(U_i^*, \mathcal{V}|_{U_i^*})$ do not change if we shrink the disk U_i . Therefore, by the exactness of (3) we have $H^n(U_i, (j)\mathcal{V})|_{U_i} = 0$ and hence

$$(4) \quad H_{x_i}^n(U_i, (j)\mathcal{V})|_{U_i} \cong H^{n-1}(U_i^*, \mathcal{V}|_{U_i^*}) \cong \begin{cases} \text{Ker}(g_i - 1), & n = 1, \\ \text{Coker}(g_i - 1), & n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

For the second isomorphism we have used $H^{n-1}(U_i^*, \mathcal{V}|_{U_i^*}) \cong H^{n-1}(\pi_1(U_i^*), V)$. Consider the long exact sequence

$$(5) \quad \cdots \rightarrow H_D^n(X, j\mathcal{V}) \rightarrow H_c^n(U, \mathcal{V}) \rightarrow H^n(U, \mathcal{V}) \rightarrow \cdots.$$

By (4), the image in $H_D^2(X, j\mathcal{V})$ of the class of a 1-cocycle $\delta : \pi_1(U) \rightarrow V$ vanishes if and only if $v_i := \delta(\alpha_i) \in \text{Im}(g_i - 1)$. The lemma follows now from the exactness of (5) and from Proposition 1.1 (ii). ■

The preceding lemma shows that the association $\delta \mapsto (v_i)$ yields an isomorphism

$$H_p^1(U, \mathcal{V}) \cong W_{\mathbf{g}} := H_{\mathbf{g}}/E_{\mathbf{g}},$$

where

$$H_{\mathbf{g}} := \{(v_1, \dots, v_r) \mid v_i \in \text{Im}(g_i - 1), \text{ relation (2) holds}\}$$

and

$$E_{\mathbf{g}} := \{(v \cdot (g_1 - 1), \dots, v \cdot (g_r - 1)) \mid v \in V\}.$$

The R -module $W_{\mathbf{g}}$ has already been defined in [26], where it is called the **braid companion** of V .

Remark 1.3: Suppose that $R = K$ is a field and that the stabilizer $V^{\pi_1(U)}$ is trivial. Then the Ogg–Shafarevic formula implies the following dimension formula:

$$\dim_K H_p^1(U, \mathcal{V}) = (r - 2) \dim_K V - \sum_{i=1}^r \dim_K \text{Ker}(g_i - 1).$$

This formula can also be verified directly using the isomorphism $H_p^1(U, \mathcal{V}) \cong W_{\mathbf{g}}$.

2. Variation of a local system

We study variations of local systems on the punctured sphere, with moving singularities. The main result is the computation of the monodromy of the parabolic cohomology of the variation. This computation is based on a natural generalization of results of Völklein [25], [26].

2.1. Let S be a connected complex manifold, and $r \geq 3$. An r -**configuration** over S consists of a smooth and proper morphism $\bar{\pi}: X \rightarrow S$ of complex manifolds together with a smooth relative divisor $D \subset X$ such that the following holds. For all $s \in S$ the fiber $X_s := \bar{\pi}^{-1}(s)$ is a Riemann surface of genus 0, and the divisor $D \cap X_s$ consists of r pairwise distinct points x_1, \dots, x_r .

Let us fix an r -configuration (X, D) over S . We set $U := X - D$ and denote by $j: U \hookrightarrow X$ the natural inclusion. Also, we write $\pi: U \rightarrow S$ for the natural projection. Choose a base point $s_0 \in S$ and set $X_0 := \bar{\pi}^{-1}(s_0)$ and $D_0 := X_0 \cap D$. Write $D_0 = \{x_1, \dots, x_r\}$ and $U_0 := X_0 - D_0 = \pi^{-1}(s_0)$. Choose a base point $x_0 \in U_0$. The projection $\pi: U \rightarrow S$ is a topological fibration and yields a short exact sequence

$$(6) \quad 1 \longrightarrow \pi_1(U_0, x_0) \longrightarrow \pi_1(U, x_0) \longrightarrow \pi_1(S, s_0) \longrightarrow 1.$$

From now on, we shall drop the base points from our notation. Let \mathcal{V}_0 be a local system of R -modules on U_0 , corresponding to a representation $\rho_0: \pi_1(U_0) \rightarrow \text{GL}(V)$, as in §1.2.

Definition 2.1: A **variation** of \mathcal{V}_0 over S is a local system \mathcal{V} of R -modules on U whose restriction to U_0 is identified with \mathcal{V}_0 . The **parabolic cohomology** of a variation \mathcal{V} is the higher direct image sheaf

$$\mathcal{W} := R^1 \bar{\pi}_*(j_* \mathcal{V}).$$

A variation \mathcal{V} of \mathcal{V}_0 corresponds to a representation $\rho: \pi_1(U) \rightarrow \text{GL}(V)$ whose restriction to $\pi_1(U_0)$ is equal to ρ_0 . By definition, the parabolic cohomology \mathcal{W} of the variation \mathcal{V} is a sheaf of R -modules on S . Locally on S , the configuration (X, D) is topologically trivial, i.e. there exists a homeomorphism $X \cong X_0 \times S$ which maps D to $D_0 \times S$. It follows immediately that \mathcal{W} is a local system with fibre

$$W := H_p^1(U_0, \mathcal{V}_0).$$

In other words, \mathcal{W} corresponds to a representation $\eta: \pi_1(S) \rightarrow \text{GL}(W)$. The following lemma provides a description of η in terms of cocycles.

LEMMA 2.2: *Let $\beta \in \pi_1(S)$ and $\delta: \pi_1(U_0) \rightarrow V$ be a parabolic 1-cocycle. We write $[\delta]$ for the class of δ in W . Let $\tilde{\beta} \in \pi_1(U)$ be a lift of β . Then $[\delta]^{\eta(\beta)} = [\delta']$, where $\delta': \pi_1(U_0) \rightarrow V$ is the 1-cocycle*

$$\alpha \mapsto \delta(\tilde{\beta}\alpha\tilde{\beta}^{-1}) \cdot \rho(\tilde{\beta}), \quad \alpha \in \pi_1(U_0).$$

Proof: We consider β as a continuous map $\beta: I := [0, 1] \rightarrow S$. Since I is simply connected, there exists a continuous family of homeomorphisms $\bar{\phi}_t: X_0 \xrightarrow{\sim} X_t := \bar{\pi}^{-1}(t)$, for $t \in I$, such that $\bar{\phi}_t(D_0) = D_t := X_t \cap D$ and such that $\bar{\phi}_0$ is the identity. Let ϕ_t denote the restriction of $\bar{\phi}_t$ to U_0 . Note that $\phi_1: U_0 \xrightarrow{\sim} U_0$ is a homeomorphism of U_0 with itself, whose homotopy class depends only on $\beta \in \pi_1(S)$. We may further assume that $\phi_1(x_0) = x_0$. Then $\tilde{\beta}: t \mapsto \phi_t(x_0)$ is a closed path in U with base point x_0 . The class of $\tilde{\beta}$ in $\pi_1(U)$ (which we also denote by $\tilde{\beta}$) is a lift of $\beta \in \pi_1(S)$. By the definition of ϕ_t we have, for all $\alpha \in \pi_1(U_0)$, the equality

$$(7) \quad \phi_1(\alpha) = \tilde{\beta}^{-1}\alpha\tilde{\beta}.$$

Since \mathcal{V} is a local system on U , there exists a unique continuous family $\psi_t: \mathcal{V}_0 \xrightarrow{\sim} \phi_t^*(\mathcal{V}|_{U_t})$ of isomorphisms of local systems on U_0 such that ψ_0 is the identity on \mathcal{V}_0 . Evaluation of ψ_t at the point $\tilde{\beta}(t) = \phi_t(x_0)$ yields a continuous family of isomorphism $\psi_t(x_0): V \xrightarrow{\sim} \mathcal{V}_{\tilde{\beta}(t)}$. This family corresponds to a trivialization of $\tilde{\beta}^*\mathcal{V}$, and we get

$$(8) \quad \rho(\tilde{\beta}) = \psi_1(x_0).$$

The pair (ϕ_t, ψ_t) induces a continuous family of isomorphisms

$$\lambda_t: W \xrightarrow{\sim} \mathcal{W}_t = H_p^1(U_t, \mathcal{V}|_{U_t}).$$

Using (7) and (8), one finds that $\lambda_1([\delta]) = [\delta']$, where

$$\delta'(\alpha) = \psi_1(\delta(\phi_1^{-1}(\alpha))) = \delta(\tilde{\beta}\alpha\tilde{\beta}^{-1}) \cdot \rho(\tilde{\beta}).$$

By definition of the representation η , we have $[\delta]^{\eta(\beta)} = \lambda_1([\delta]) = [\delta']$. This completes the proof of the lemma. ■

Remark 2.3: With the notation introduced above, let \mathcal{V}_0 be the local system of R -modules, corresponding to a representation $\rho_0: \pi_1(U_0) \rightarrow \text{GL}(V)$.

- (1) A necessary condition for the existence of a variation of \mathcal{V}_0 over S is the following. For every element $\tilde{\beta} \in \pi_1(U, x_0)$ there exists an element $g \in \text{GL}(V)$ such that

$$\rho_0(\tilde{\beta}\alpha\tilde{\beta}^{-1}) = g\rho_0(\alpha)g^{-1}$$

holds for all $\alpha \in \pi_1(U_0)$.

- (2) Suppose that S is a smooth affine curve, and that (i) holds. Then there exists a variation \mathcal{V} of \mathcal{V}_0 over S . This follows easily from the fact that $\pi_1(S)$ is a free group.
- (3) Suppose, moreover, that R is an integral domain and that \mathcal{V}_0 is irreducible. If \mathcal{V}' is another variation of \mathcal{V}_0 over S , then there exists a local system \mathcal{L} of rank one on S such that $\mathcal{V}' \cong \mathcal{V} \otimes_R \pi^* \mathcal{L}$. Let \mathcal{W} (resp. \mathcal{W}') denote the parabolic cohomology of \mathcal{V} (resp. of \mathcal{V}'). By the projection formula we have

$$\mathcal{W}' \cong R^1 \bar{\pi}_*(j_* \mathcal{V} \otimes \bar{\pi}^* \mathcal{L}) \cong \mathcal{W} \otimes \mathcal{L}.$$

Therefore, the projective representation associated to \mathcal{W} ,

$$\lambda: \pi_1(S) \longrightarrow \text{PGL}(W),$$

is uniquely determined by \mathcal{V}_0 .

2.2 THE ARTIN BRAID GROUP AND THE COCYCLES $\Phi(\mathbf{g}, \beta)$. Let $D_0 \subset \mathbb{C}$ be a set of r distinct complex numbers and set $U_0 := \mathbb{P}^1_{\mathbb{C}} - D_0$. We choose a marking κ of $(\mathbb{P}^1_{\mathbb{C}}, D_0)$ which maps ∞ into the upper half plane; see §1.2. The choice of κ induces a presentation of $\pi_1(U_0, \infty)$, with generators $\alpha_1, \dots, \alpha_r$ and relation $\prod_i \alpha_i = 1$.

Let

$$\mathcal{O}_r := \{D \subset \mathbb{C} \mid |D| = r\}$$

denote the set of all subsets of \mathbb{C} of cardinality r . There is a standard way to identify \mathcal{O}_r with the complement in \mathbb{C}^r of the ‘discriminant locus’: given $D \in \mathcal{O}_r$, the corresponding point in \mathbb{C}^r is (c_1, \dots, c_r) , where $X^r + c_1 X^{r-1} + \dots + c_r = \prod_{z \in D} (X - z)$. The fundamental group $A_r := \pi_1(\mathcal{O}_r, D_0)$ is called the **Artin braid group** on r strands. The group A_r has $r - 1$ standard generators $\beta_1, \dots, \beta_{r-1}$ with relations

$$\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}, \quad \beta_i \beta_j = \beta_j \beta_i,$$

for $1 \leq i < r$ and $i < j - 1 < r - 1$. The element β_i is represented by the path $t \mapsto \{x_1, \dots, \delta_i^-(t), \delta_i^+(t), \dots, x_r\}$, where δ_i^+ (resp. δ_i^-) is a path from x_{i+1} to x_i through the inverse image under κ of the upper half plane (resp. from x_i to x_{i+1} through the inverse image of the lower half plane).

Define

$$\mathcal{O}_{r,1} := \{(D, x) \mid D \in \mathcal{O}_r, x \in \mathbb{P}_{\mathbb{C}}^1 - D\}.$$

The natural projection $\mathcal{O}_{r,1} \rightarrow \mathcal{O}_r$ is a topological fibration with fiber U_0 , and admits a section $D \mapsto (D, \infty)$. It yields a split exact sequence of fundamental groups

$$(9) \quad 1 \rightarrow \pi_1(U_0, \infty) \longrightarrow \pi_1(\mathcal{O}_{r,1}, (D_0, \infty)) \longrightarrow A_r \rightarrow 1.$$

We may identify A_r with its image in $\pi_1(\mathcal{O}_{r,1})$ under the splitting induced from the section $D \mapsto (D, \infty)$. Then A_r acts, by conjugation, on $\pi_1(U_0, \infty)$. We have the following well-known formulas for this action:

$$(10) \quad \beta_i^{-1} \alpha_j \beta_i = \begin{cases} \alpha_i \alpha_{i+1} \alpha_i^{-1}, & \text{for } j = i, \\ \alpha_i, & \text{for } j = i + 1, \\ \alpha_j, & \text{otherwise.} \end{cases}$$

Let R be a commutative ring and V a free R -module of finite rank. Define

$$\mathcal{E}_r := \{\mathbf{g} = (g_1, \dots, g_r) \mid g_i \in \text{GL}(V), \prod_i g_i = 1\}.$$

An element $\mathbf{g} \in \mathcal{E}_r$ corresponds to a representation $\rho_0: \pi_1(U_0) \rightarrow \text{GL}(V)$ (set $\rho_0(\alpha_i) := g_i$) and hence to a local system \mathcal{V}_0 on U_0 . Given $\beta \in A_r$, we set

$$(11) \quad \rho_0^\beta(\alpha) := \rho_0(\beta \alpha \beta^{-1})$$

and call the local system \mathcal{V}_0^β corresponding to the representation ρ_0^β the **twist** of \mathcal{V}_0 by β . We denote by \mathbf{g}^β the element of \mathcal{E}_r corresponding to ρ_0^β . This defines an action of A_r on \mathcal{E}_r , from the right. From (10), we get the following formula for the effect of the standard generators β_i on \mathcal{E}_r :

$$(12) \quad \mathbf{g}^{\beta_i} = (g_1, \dots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \dots, g_r).$$

Given $\mathbf{g} \in \mathcal{E}_r$, we have defined in §1.3 the R -module

$$H_{\mathbf{g}} = \{(v_1, \dots, v_r) \mid v_i \in \text{Im}(g_i - 1), \text{ relation (2) holds}\}.$$

An element $(v_1, \dots, v_r) \in H_{\mathbf{g}}$ corresponds to a parabolic 1-cocycle $\delta: \pi_1(U_0) \rightarrow V$, determined by $\delta(\alpha_i) = v_i$, for $i = 1, \dots, r$. Here the $\pi_1(U_0)$ -module structure

on V is induced by \mathfrak{g} . We say that δ is a parabolic 1-cocycle with respect to \mathfrak{g} . Given $\beta \in A_r$ and $\alpha \in \pi_1(U_0)$, set

$$\delta^\beta(\alpha) := \delta(\beta\alpha\beta^{-1}).$$

One easily checks that $\delta^\beta: \pi_1(U_0) \rightarrow V$ is a parabolic 1-cocycle with respect to \mathfrak{g}^β . Moreover, the association $\delta \mapsto \delta^\beta$ defines an R -linear map

$$\Phi(\mathfrak{g}, \beta): H_{\mathfrak{g}} \longrightarrow H_{\mathfrak{g}^\beta}.$$

In order to maintain compatibility with our convention of ‘acting from the right’, we write $(v_i)^{\Phi(\mathfrak{g}, \beta)}$ for the image of $(v_i) \in H_{\mathfrak{g}}$ under $\Phi(\mathfrak{g}, \beta)$. Using (10) and the fact that δ is a 1-cocycle with respect to \mathfrak{g} , we get

$$(13) \quad (v_1, \dots, v_r)^{\Phi(\mathfrak{g}, \beta)} = (v_1, \dots, v_{i+1}, \underbrace{v_{i+1}(1 - g_{i+1}^{-1}g_i g_{i+1}) + v_i g_{i+1}}_{(i+1)\text{th entry}}, \dots, v_r).$$

Moreover, we have the ‘cocycle rule’

$$(14) \quad \Phi(\mathfrak{g}, \beta) \cdot \Phi(\mathfrak{g}^\beta, \beta') = \Phi(\mathfrak{g}, \beta\beta').$$

(The product on the left hand side of (14) is defined as the function from $H_{\mathfrak{g}}$ to $H_{\mathfrak{g}^{\beta\beta'}}$ obtained by first applying $\Phi(\mathfrak{g}, \beta)$ and then $\Phi(\mathfrak{g}^\beta, \beta')$.)

The submodule

$$E_{\mathfrak{g}} := \{(v \cdot (g_1 - 1), \dots, v \cdot (g_r - 1) \mid v \in V\}$$

of $H_{\mathfrak{g}}$ corresponds to 1-cocycles δ which are coboundaries. It is easy to see that $\Phi(\mathfrak{g}, \beta)$ maps $E_{\mathfrak{g}}$ into $E_{\mathfrak{g}^\beta}$ and therefore induces an isomorphism

$$\bar{\Phi}(\mathfrak{g}, \beta) : W_{\mathfrak{g}} := H_{\mathfrak{g}}/E_{\mathfrak{g}} \xrightarrow{\sim} W_{\mathfrak{g}^\beta}.$$

One can compute $\bar{\Phi}(\mathfrak{g}, \beta)$ explicitly for all $\beta \in A_r$ using (13) and (14), provided that β is given as a word in the standard generators β_i . Moreover, this computation can easily be implemented on a computer.

Given $\mathfrak{g} \in \mathcal{E}_r$ and $h \in \text{GL}(V)$, we set

$$\mathfrak{g}^h := (h^{-1}g_1h, \dots, h^{-1}g_rh),$$

and we define an isomorphism

$$\Psi(\mathfrak{g}, h): \begin{cases} H_{\mathfrak{g}^h} & \xrightarrow{\sim} & H_{\mathfrak{g}} \\ (v_1, \dots, v_r) & \longmapsto & (v_1 \cdot h, \dots, v_r \cdot h). \end{cases}$$

It is clear that $\Psi(\mathfrak{g}, h)$ maps $E_{\mathfrak{g}^h}$ to $E_{\mathfrak{g}}$ and therefore induces an isomorphism $\bar{\Psi}(\mathfrak{g}, h): W_{\mathfrak{g}^h} \xrightarrow{\sim} W_{\mathfrak{g}}$.

2.3 EXPLICIT COMPUTATION OF THE MONODROMY. Let us go back to the situation of §2.1: we are given an r -configuration (X, D) over a connected complex manifold S . We have also chosen a base point $s_0 \in S$. As usual, we set $U := X - D$, and denote by U_0 the fiber of $U \rightarrow S$ over s_0 .

Definition 2.4: An **affine frame** for the configuration (X, D) is an isomorphism of complex manifolds $\lambda: X \cong \mathbb{P}_S^1$, compatible with the projection to S , such that $\lambda(D)$ either contains or is disjoint from $\{\infty\} \times S$.

In this subsection, we shall assume that there exists an affine frame for (X, D) , and we use it to identify X with \mathbb{P}_S^1 . We remark that there exist configurations (X, D) which do not admit an affine frame (e.g. because $X \not\cong \mathbb{P}_S^1$). It seems, however, that such examples have no practical relevance for the problems this paper is about.

By the nature of Definition 2.4, there are two cases to consider. Suppose first that D is disjoint from $\{\infty\} \times S$. Then D gives rise to a map $p: S \rightarrow \mathcal{O}_r$ which sends $s \in S$ to the fiber of $D \subset \mathbb{A}_S^1 \rightarrow S$ over s . Set $D_0 := p(s_0) \subset \mathbb{C}$. Choose a marking κ of $(\mathbb{P}_{\mathbb{C}}^1, D_0)$. We will use κ to identify the fundamental group $\pi_1(\mathcal{O}_r, D_0)$ with the Artin braid group A_r , as in the previous subsection. Let $\varphi: \pi_1(S, s_0) \rightarrow A_r$ denote the group homomorphism induced by p . The exact sequence

$$(15) \quad 1 \rightarrow \pi_1(U_0, \infty) \longrightarrow \pi_1(U, (\infty, s_0)) \longrightarrow \pi_1(S, s_0) \rightarrow 1$$

of the fibration $U \rightarrow S$ can be identified with the pullback of the sequence (9) along φ . Using the splitting of (15) coming from the ∞ -section, we will consider $\pi_1(S)$ as a subgroup of $\pi_1(U)$. By construction, the action of $\pi_1(S)$ on $\pi_1(U_0, \infty)$ factors through the map φ and is given by the formulas (10).

Now suppose that D contains the section $\{\infty\} \times S$. We denote by $\pi_1(U_0, \infty)$ the fundamental group of U_0 with ∞ as ‘tangential base point’. More precisely, consider subsets of $U_0 \subset \mathbb{C}$ of the form $\Omega_t = \{z \in \mathbb{C} \mid |z| > t, z \notin (-\infty, 0)\}$, for $t \gg 0$. The fundamental group $\pi_1(U_0, \Omega_t)$ is independent of t , up to *canonical* isomorphism, so we may define $\pi_1(U_0, \infty) := \lim \pi_1(U_0, \Omega_t)$. With this convention, the sequence (15) is still well defined and admits a canonical section. In fact, the fibration $U \rightarrow S$ admits a section $\xi: S \rightarrow U$, unique up to homotopy, such that for all $s \in S$ we have $\xi(s) \in \Omega_t \subset U_s$, for some $t > 0$. As in the first case, we will identify $\pi_1(S)$ with the image of this section.

The **Hurwitz braid group** B_r is defined as the fundamental group of the set

$$\mathcal{U}_r = \{D \subset \mathbb{P}_{\mathbb{C}}^1 \mid |D| = r\},$$

with base point D_0 . The natural map $\mathcal{O}_r \rightarrow \mathcal{U}_r$ identifies B_r with the quotient of A_r by the relation

$$\beta_1\beta_2 \cdots \beta_{r-1}^2 \cdots \beta_2\beta_1 = 1.$$

The configuration (X, D) induces a map $p: S \rightarrow \mathcal{U}_r$ and a homomorphism $\varphi: \pi_1(S) \rightarrow B_r$. If $\{\infty\} \times S \subset D$, then the image of φ is contained in the subgroup of B_r generated by the first $r-2$ standard braids $\beta_1, \dots, \beta_{r-2}$, which is isomorphic to A_{r-1} . Moreover, just as in the first case, the action of $\pi_1(S, s_0)$ on $\pi_1(U_0, \infty)$ by conjugation factors through the map φ and is given by the formulas (10). From now on, we will treat both cases of Definition 2.4 simultaneously.

Let \mathcal{V}_0 be a local system of free R -modules on $U_0 = \mathbb{P}^1(\mathbb{C}) - D_0$, corresponding to a representation $\rho_0: \pi_1(U_0) \rightarrow \text{GL}(V)$. A variation of \mathcal{V}_0 over S corresponds, by definition, to a representation $\rho: \pi_1(U) \rightarrow \text{GL}(V)$ whose restriction to $\pi_1(U_0)$ equals ρ_0 . Obviously, ρ is uniquely determined by its restriction to $\pi_1(S)$, which we denote by $\chi: \pi_1(S) \rightarrow \text{GL}(V)$. Then

$$(16) \quad \rho_0(\gamma\alpha\gamma^{-1}) = \chi(\gamma)\rho_0(\alpha)\chi(\gamma)^{-1}$$

holds for all $\alpha \in \pi_1(U_0)$ and $\gamma \in \pi_1(S)$. With $\mathbf{g} \in \mathcal{E}_r$ corresponding to ρ_0 (via the choice of the marking κ), this is equivalent to

$$(17) \quad \mathbf{g}^{\varphi(\gamma)} = \mathbf{g}^{\chi(\gamma)^{-1}}.$$

Let \mathcal{W} be the parabolic cohomology of \mathcal{V} and $\eta: \pi_1(S) \rightarrow \text{GL}(W_{\mathbf{g}})$ the corresponding representation (here we identify the fiber of \mathcal{W} at s_0 with the R -module $W_{\mathbf{g}} = H_{\mathbf{g}}/E_{\mathbf{g}}$, see the previous subsection).

THEOREM 2.5: *For all $\gamma \in \pi_1(S)$ we have*

$$\eta(\gamma) = \bar{\Phi}(\mathbf{g}, \varphi(\gamma)) \cdot \bar{\Psi}(\mathbf{g}, \chi(\gamma)),$$

where $\bar{\Phi}(\mathbf{g}, \beta): W_{\mathbf{g}} \xrightarrow{\sim} W_{\mathbf{g}^\beta}$ and $\bar{\Psi}(\mathbf{g}, h): W_{\mathbf{g}^h} \rightarrow W_{\mathbf{g}}$ are the isomorphisms defined in §2.2.

Proof: Straightforward, using Lemma 2.2, the definition of $\bar{\Phi}(\mathbf{g}, \beta)$ and $\bar{\Psi}(\mathbf{g}, h)$, and (17). ■

Remark 2.6: Consider the case where D contains the section $\infty \times S$. Using similar arguments as above one can show that for any choice of base point of U_0 (not necessarily the tangential base point ∞) one obtains a splitting of Sequence (15) which is compatible with the map to the sequence corresponding to (9).

(For fibrations of plane curve complement this is made explicit in [8], Thm. 2.2.) In this situation, the statement of Theorem 2.5 carries over verbatim.

3. Étale local systems

We transfer the situation considered in the first two sections into the étale world, and we state a comparison theorem. We also prove a theorem which is useful to bound the field of linear moduli of an (étale) local system obtained as the parabolic cohomology of a variation.

3.1 RECALL. In this section, we fix a prime number l and a finite extension K/\mathbb{Q}_l . We denote by R one of the following rings: (a) $R := K$, (b) $R := \mathcal{O}_K$, the ring of integers of K , or (c) $R := \mathcal{O}_K/\ell^m$, where ℓ is the prime ideal of \mathcal{O}_K .

Let k be a field of characteristic 0 and X a smooth, geometrically irreducible scheme over k . Also, let $x: \text{Spec } \bar{k} \rightarrow X$ be a geometric point. We denote by $\pi_1(X) = \pi_1(X, x)$ the algebraic fundamental group of X with base point x .

An **étale local system** \mathcal{V} of R -modules on X is, by definition, a *locally constant and constructible sheaf of R -modules* [19] whose stalks are free R -modules of finite rank. In case $R = K$, this is also called a **lisse ℓ -adic sheaf** [15]. It is a standard fact that \mathcal{V} corresponds to a continuous representation

$$\rho: \pi_1(X, x) \longrightarrow \text{GL}(V),$$

where $V := \mathcal{V}_x$ is the stalk of \mathcal{V} at x .

Now suppose that $k \subset \mathbb{C}$ is a subfield of the complex numbers. The set of \mathbb{C} -rational points of X has a canonical structure of a complex manifold, which we denote by X^{an} . Moreover, there is a functor $\mathcal{F} \mapsto \mathcal{F}^{\text{an}}$ from sheaves (of abelian groups) on $X_{\text{ét}}$ to sheaves on X^{an} , called **analytification** (see e.g. [13], §I.11). If \mathcal{V} is an étale local system on X corresponding to a representation $\rho: \pi_1(X, x) \rightarrow \text{GL}(V)$, then the analytification \mathcal{V}^{an} of \mathcal{V} is the local system corresponding to the composition of ρ with the natural homomorphism $\pi_1^{\text{top}}(X^{\text{an}}, x) \rightarrow \pi_1(X, x)$.

3.2 PARABOLIC COHOMOLOGY OF AN ÉTALE LOCAL SYSTEM. Let k be a field of characteristic 0 and let S be a smooth, affine and geometrically connected variety over k . Let (X, D) be an r -*configuration* over S . By this we mean that $\bar{\pi}: X \rightarrow S$ is a proper smooth curve of genus 0 and $D \subset X$ is a smooth relative divisor of relative degree r (compare with §2.1). We denote by $j: U := X - D \hookrightarrow X$ the inclusion and by $\pi: U \rightarrow S$ the natural projection. We fix a k -rational point s_0 on S as a base point. We write $U_0 := \pi^{-1}(s_0)$ for the fiber of π over s_0 and we choose a geometric point $x_0: \text{Spec } \bar{k} \rightarrow U_0$ as base point.

Definition 3.1: Let \mathcal{V}_0 be an étale local system of R -modules on U_0 . A **variation** of \mathcal{V}_0 over S is an étale local system \mathcal{V} on U whose restriction to U_0 is equal to \mathcal{V}_0 . The **parabolic cohomology** of the variation \mathcal{V} is the sheaf of R -modules on $S_{\text{ét}}$

$$\mathcal{W} := R^1 \bar{\pi}_*(j_* \mathcal{V}).$$

See [19].

THEOREM 3.2: *Suppose that $k \subset \mathbb{C}$.*

- (i) \mathcal{W} is an étale local system of R -modules.
- (ii) There is a natural isomorphism of local systems of R -modules on X^{an}

$$\mathcal{W}^{\text{an}} \xrightarrow{\sim} R^1 \bar{\pi}_*(j_* \mathcal{V}^{\text{an}}),$$

functorial in \mathcal{V} .

Proof: Using standard arguments (see e.g. [13], §12), one reduces the claim to the case $R = \mathcal{O}_K/\ell^m$. Let \mathcal{F} be a constructible sheaf of R -modules on X . By the comparison theorem between étale and singular cohomology there is a natural isomorphism of sheaves on S^{an}

$$(R^1 \bar{\pi}_* \mathcal{F})^{\text{an}} \xrightarrow{\sim} R^1 \bar{\pi}_*(\mathcal{F}^{\text{an}}).$$

(See e.g. [13], Theorem 11.6, for the case where $k = \mathbb{C}$. The general case follows immediately, using the Proper Base Change Theorem, [13], Theorem 6.1.) It is easy to see (e.g. using [13], Proposition 11.4) that $(j_* \mathcal{V})^{\text{an}} = j_*(\mathcal{V}^{\text{an}})$. Therefore, Part (ii) of the theorem follows from the comparison theorem. By [13], Theorem 8.10, the sheaf $\mathcal{W} = R^1 \bar{\pi}_*(j_* \mathcal{V})$ is constructible. But we have just proved that \mathcal{W}^{an} is locally constant, which shows that \mathcal{W} is locally constant as well. This finishes the proof of the theorem. ■

Remark 3.3: For us, the important consequence of Theorem 3.2 is that we can compute the geometric monodromy of \mathcal{W} using the methods of the previous section. Let us make this point more explicit. In the situation introduced above, let $\rho: \pi^1(U, x_0) \rightarrow \text{GL}(V)$ denote the representation of the algebraic fundamental group of U corresponding to the étale local system \mathcal{V} . Let $\rho^{\text{geo}}: \pi_1(U_{\bar{k}}, x_0) \rightarrow \text{GL}(V)$ (resp. $\rho^{\text{top}}: \pi_1^{\text{top}}(U^{\text{an}}, x_0) \rightarrow \text{GL}(V)$) denote the restriction of ρ to the geometric (resp. to the topological) fundamental group. We remark that ρ and ρ^{geo} are continuous morphisms of profinite groups, whereas ρ^{top} is simply a group homomorphism. Furthermore, ρ^{geo} is uniquely determined by ρ^{top} because $\pi_1(U_{\bar{k}}, x_0)$ is the profinite completion of $\pi_1^{\text{top}}(U^{\text{an}}, x_0)$.

By Part (i) of Theorem 3.2, the parabolic cohomology \mathcal{W} of \mathcal{V} corresponds to a representation $\eta: \pi_1(S) \rightarrow \mathrm{GL}(W)$. Let η^{geo} and η^{top} be its geometric and topological version. Then Part (ii) of Theorem 3.2 says that we have two ways to describe η^{top} : (a) as the restriction of η to $\pi_1^{\mathrm{top}}(S^{\mathrm{an}})$, and (b) as the monodromy representation of $\pi_1^{\mathrm{top}}(S^{\mathrm{an}})$ on the singular parabolic cohomology of the local system $\mathcal{V}^{\mathrm{an}}$ (which can be computed using Theorem 2.5).

3.3 THE FIELD OF LINEAR MODULI. The results of this subsection will not be used in the rest of this paper. As in the previous subsection, S denotes a smooth and geometrically connected k -variety and (X, D) an r -configuration over S . We assume that $k \subset \mathbb{C}$ and denote by \bar{k} the algebraic closure of k inside \mathbb{C} .

Unlike in the previous subsection, \mathcal{V}_0 is now an étale local system of R -modules on the *geometric* fibre $U_{0, \bar{k}} := U_0 \otimes \bar{k}$, and \mathcal{V} denotes a variation of \mathcal{V}_0 over $S_{\bar{k}} := S \otimes_k \bar{k}$. Let \mathcal{W} be the parabolic cohomology of \mathcal{V} . By construction, \mathcal{W} is an étale local system of R -modules on $S_{\bar{k}}$. For $\sigma \in \mathrm{Gal}(\bar{k}/k)$, we denote by $\tilde{\sigma}_{U_0}: U_{0, \bar{k}} \xrightarrow{\sim} U_{0, \bar{k}}$ the semi-linear automorphism corresponding to σ and the k -model U_0 . The **twist of \mathcal{V}_0 by σ** (with respect to the k -model U_0) is the étale local system $\mathcal{V}_0^\sigma := \tilde{\sigma}_{U_0}^* \mathcal{V}_0$.

Definition 3.4: We say that k is a **field of linear moduli** for \mathcal{V}_0 if the étale local system \mathcal{V}_0^σ is isomorphic to \mathcal{V}_0 , for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$. We say that k is a **field of projective moduli** for \mathcal{V}_0 if for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$ there exists an étale local system \mathcal{L}_σ of rank one such that $\mathcal{V}_0^\sigma \cong \mathcal{V}_0 \otimes_R \mathcal{L}_\sigma$. Similarly, one defines the notion of ‘field of linear/projective moduli’ for the local systems \mathcal{V} and \mathcal{W} .

THEOREM 3.5:

- (i) *If k is a field of linear moduli for \mathcal{V} , then it is also a field of linear moduli for \mathcal{W} .*
- (ii) *Suppose that $R = \mathcal{O}_K$ or $R = K$, and that \mathcal{V}_0 is irreducible. Then if k is a field of projective moduli for \mathcal{V}_0 , it is also a field of projective moduli for \mathcal{W} . Moreover, the projective representation $\lambda^{\mathrm{geo}}: \pi_1(S_{\bar{k}}) \rightarrow \mathrm{PGL}(W)$ associated to \mathcal{W} extends to a projective representation $\lambda: \pi_1(S) \rightarrow \mathrm{PGL}(W)$.*

Proof: Let $\sigma \in \Gamma_{k_0}$ with $\mathcal{V}^\sigma \cong \mathcal{V}$. Using the Proper Base Change Theorem we get

$$\mathcal{W}^\sigma = \hat{\sigma}_S^* R^1 \bar{\pi}_{\bar{k},*} (j_{\bar{k},*} \mathcal{V}) = R^1 \bar{\pi}_{\bar{k},*} (j_{\bar{k},*} \mathcal{V}^\sigma) \cong \mathcal{W}.$$

This proves (i). The proof of (ii) is a combination of the preceding argument and Remark 2.3 (iii). ■

Remark 3.6: Theorem 3.5 can be used to give a new proof of the main result of [27] (which essentially states that the braid companion functor preserves the field of linear moduli). In the rest of the paper, we will not use Theorem 3.5 and we will not need the concept ‘field of linear moduli’. The point is that in our main example in §5, the variation \mathcal{V} is already known to be defined over \mathbb{Q} , which means that the resulting local system \mathcal{W} is defined over \mathbb{Q} as well, by construction. In fact, this seems to be the case for all known applications of these and similar methods to the Regular Inverse Galois Problem. Nevertheless, the authors think that Theorem 3.5 may be useful for future applications.

4. Local systems on Hurwitz spaces

A finite Galois cover $f: Y \rightarrow \mathbb{P}^1$ together with a representation $G \hookrightarrow \mathrm{GL}_n(K)$ of its Galois group corresponds to a local system on \mathbb{P}^1 with finite monodromy. Therefore, a representation $G \hookrightarrow \mathrm{GL}_n(K)$ gives rise to a variation of local systems on a certain Hurwitz space H . Since Hurwitz spaces are algebraic varieties, the parabolic cohomology of this variation corresponds to a Galois representation of the function field of H . In case H is a rational variety, this has potential applications to the Regular Inverse Galois Problem.

A special case of this construction is discussed in Section 5. In the present section, we give an outline of the general construction. We do not, however, work in the greatest possible generality.

4.1. In this section we fix a finite group G and an integer $r \geq 3$. The letter S will always denote a scheme over $\mathrm{Spec}(\mathbb{Q})$.

Definition 4.1: Let $X \rightarrow S$ be a smooth projective curve of genus 0. A **G -cover of X with r branch points** is a finite morphism $f: Y \rightarrow X$, together with an isomorphism $G \cong \mathrm{Aut}(Y/X)$, such that the following holds. First, f is tamely ramified along a smooth relative divisor $D \subset X$ with constant degree $r = \deg(D/S)$, and étale over $U := X - D$. Second, for each geometric point $s: \mathrm{Spec} k \rightarrow S$, the pullback $f_s: Y_s \rightarrow X_s$ is a G -Galois cover branched at $D_s := D \cap X_s$ (in particular, Y_s is connected). Let $X_1 \rightarrow S$ and $X_2 \rightarrow S$ be two curves of genus 0 over S and $f_1: Y_1 \rightarrow X_1$ and $f_2: Y_2 \rightarrow X_2$ two G -covers with r branch points. We say that f_1 and f_2 are **isomorphic** if there exists isomorphism of S -schemes $\phi: X_1 \xrightarrow{\sim} X_2$ and a G -equivariant isomorphism $\psi: Y_1 \xrightarrow{\sim} Y_2$ such that $\phi \circ f_1 = f_2 \circ \psi$.

By the result of [14], [4] and [28], there exists a certain \mathbb{Q} -scheme, denoted by

$$H_r^{\mathrm{red}}(G),$$

which is a coarse moduli space for G -Galois covers of curves of genus 0 with r branch points. In particular, to each G -Galois cover $f: Y \rightarrow X$ of a genus zero curve X over a \mathbb{Q} -scheme S , we can associate a map $\varphi_f: S \rightarrow H_r^{\text{red}}(G)$, called the **classifying map** for f . The association $f \mapsto \varphi_f$ is functorial in S . For $S = \text{Spec } k$, where k is an algebraically closed field of characteristic 0, it induces a bijection between isomorphism classes of G -covers of curves of genus 0 with r branch points, defined over k , and k -rational points on $H_r^{\text{red}}(G)$. The scheme $H_r^{\text{red}}(G)$ is called the **reduced inner Hurwitz space**. It is a normal and affine variety over \mathbb{Q} , of dimension $r - 3$. We point out that $H_r^{\text{red}}(G)$ is in general not geometrically connected.

Set $H := H_r^{\text{red}}(G)$. A **versal family** over H is a G -cover $f: Y \rightarrow X$ defined over a \mathbb{Q} -scheme S such that S is normal and the classifying map $\varphi_f: S \rightarrow H$ is finite, flat and generically étale. A versal family f is said to **cover** the Hurwitz space H if the classifying map φ_f is surjective onto H . The cover f is called **universal** if, moreover, for every S -scheme T the pullback map induced by f (from $\text{Hom}(T, S)$ to the set of isomorphism classes of G -covers defined over T) is an isomorphism. Note that the classifying map of a universal family is automatically an isomorphism.

For every G and r there always exists a versal family of G -covers which covers the Hurwitz space H . However, a universal family exists only under certain extra assumptions.

4.2. Let us now fix the following objects:

- a versal family $f: Y \rightarrow X$ of G -covers with r branch points, defined over a connected and normal \mathbb{Q} -scheme S , and
- a faithful and irreducible linear representation $G \hookrightarrow \text{GL}_n(K)$, with coefficients in a number field K .

Let k be the field of definition of S , i.e. the integral closure of \mathbb{Q} in the function field of S . Then k is number field and S is a geometrically irreducible k -variety.

Choose an integer N such that $G \subset \text{GL}_n(K)$ is contained in $\text{GL}_n(R)$, where $R = \mathcal{O}_K[1/N]$. Let $D \subset X$ be the branch locus of $f: Y \rightarrow X$, and set $U := X - D$. The G -cover $f: Y \rightarrow X$ gives rise to a surjective group homomorphism $\pi_1(U) \rightarrow G$. We denote by

$$\rho: \pi_1(U) \longrightarrow \text{GL}_n(R)$$

the composition of this homomorphism with the injection $G \hookrightarrow \text{GL}_n(R)$.

We shall write $\rho^{\text{geo}}: \pi_1(U_{\overline{\mathbb{Q}}}) \rightarrow \text{GL}_n(R)$ (resp. $\rho^{\text{top}}: \pi_1^{\text{top}}(U_{\mathbb{C}}) \rightarrow \text{GL}_n(R)$) for the restriction of ρ to the geometric fundamental group of U (resp. to the

topological fundamental group of the analytic space associated to U). Let \mathcal{V}^{an} denote the local system of R -modules on $U_{\mathbb{C}}$ corresponding to ρ^{top} . Also, let

$$\mathcal{W}^{\text{an}} := R^1 \bar{\pi}_*(j_* \mathcal{V}^{\text{an}})$$

be the parabolic cohomology of \mathcal{V}^{an} . Recall that \mathcal{W}^{an} is a local system of R -modules corresponding to a representation $\eta^{\text{top}}: \pi_1^{\text{top}}(S_{\mathbb{C}}) \rightarrow \text{GL}(W)$.

On the other hand, for each prime ideal \mathfrak{p} of K which is prime to N we let

$$\rho_{\mathfrak{p}}: \pi_1(U) \longrightarrow \text{GL}_n(\mathcal{O}_{\mathfrak{p}})$$

denote the p -adic representation induced by ρ . It corresponds to an étale local system $\mathcal{V}_{\mathfrak{p}}$ of $\mathcal{O}_{\mathfrak{p}}$ -modules on U . Again we can form the (parabolic) higher direct image of $\mathcal{V}_{\mathfrak{p}}$,

$$\mathcal{W}_{\mathfrak{p}} := R^1 \bar{\pi}_*(j_* \mathcal{V}_{\mathfrak{p}}),$$

which is an étale local system of $\mathcal{O}_{\mathfrak{p}}$ -modules, corresponding to a representation $\eta_{\mathfrak{p}}: \pi_1(S) \rightarrow \text{GL}(W_{\mathfrak{p}})$. It follows from Theorem 3.2 (see also Remark 3.3) that there is a canonical isomorphism $W_{\mathfrak{p}} \cong W \otimes_R \mathcal{O}_{\mathfrak{p}}$ such that the following diagram commutes:

$$\begin{CD} \pi_1^{\text{top}}(S_{\mathbb{C}}) @>\eta^{\text{top}}>> \text{GL}(W) \\ @VVV @VVV \\ \pi_1(S) @>\eta_{\mathfrak{p}}>> \text{GL}(W_{\mathfrak{p}}). \end{CD}$$

In particular, the image of $\eta_{\mathfrak{p}}^{\text{geo}}$ is equal to the topological closure of the image of η^{top} .

Remark 4.2:

- (i) Let $s \in S(k)$ be a k -rational point. The fiber of $\mathcal{W}_{\mathfrak{p}}$ at x corresponds to a continuous Galois representation $\eta_{x,\mathfrak{p}}: \text{Gal}(\bar{k}/k) \rightarrow \text{GL}(W_{\mathfrak{p}})$. One can show that the family $(\eta_{x,\mathfrak{p}})_{\mathfrak{p}}$ forms a *strictly compatible system of Galois representations of weight one* (see [9], Proposition 5.4.5 for a more general statement).
- (ii) Fix \mathfrak{p} and suppose that S is a rational variety over k . In many cases one can use Hilbert's Irreducibility theorem to conclude that there exists infinitely many k -rational points $x \in S(k)$ such that the image of the Galois representation $\eta_{x,\mathfrak{p}}$ contains the image of the geometric monodromy representation $\eta_{\mathfrak{p}}^{\text{geo}}$ (which is independent of x and can in principal be calculated by the methods of Section 2). See Section 5 for an example of this phenomenon.

4.3. In the context of the Inverse Galois Problem, Remark 4.2 shows that it is desirable to produce examples of versal families of G -covers defined over a \mathbb{Q} -rational variety S , and to compute the image of the geometric monodromy representation η^{geo} explicitly. The basic tool for doing this is, as in Section 2, the braid action. To simplify the exposition, and because this is the only case that we will need later on, we assume (after the next two paragraphs) that $r = 4$.

With G and r as before, set

$$\mathcal{E}_r(G) := \{ \mathbf{g} = (g_1, \dots, g_r) \mid g_i \neq 1, G = \langle g_i \rangle, \prod_i g_i = 1 \}.$$

An element \mathbf{g} of this set is called a **generating system** of length r for the group G . The group G acts on the set $\mathcal{E}_r(G)$ by simultaneous conjugation. We write $\text{Ni}_r(G)$ for the sets of orbits of this action. Elements of $\text{Ni}_r(G)$ are called **Nielsen classes** and written as $[\mathbf{g}]$, with $\mathbf{g} \in \mathcal{E}_r(G)$.

The Artin braid group A_r acts on the set $\mathcal{E}_r(G)$ from the right, in a standard way; see (12). This action extends to an action of its quotient B_r , the Hurwitz braid group. By abuse of notation, we denote the image of the standard generator $\beta_i \in A_r$ in B_r by the same name.

From now on, we assume that $r = 4$. The elements $\beta_1\beta_3^{-1}$ and $(\beta_1\beta_2\beta_3)^2$ generate a normal subgroup $Q \triangleleft B_4$, isomorphic to the Klein 4-group. The quotient $\bar{B}_4 := B_4/Q$ is called the **mapping class group**. The set of Q -orbits of $\text{Ni}_4(G)$ is denoted by $\text{Ni}_4^{\text{red}}(G)$. Elements of this set are called **reduced Nielsen classes**, and are written as $[\mathbf{g}]^{\text{red}}$. The action of B_4 on $\text{Ni}_4(G)$ descends to an action of the mapping class group \bar{B}_4 on $\text{Ni}_4^{\text{red}}(G)$.

Let $\mathbf{C} = (C_1, \dots, C_4)$ be an ordered r -tuple of conjugacy classes of the group G . We say that $\mathbf{g} \in \mathcal{E}_4(G)$ has **type \mathbf{C}** if there exist an integer n , prime to the order of G , and a permutation $\sigma \in S_4$ such that $g_i^n \in C_{\sigma(i)}$ for $i = 1, \dots, r$. The subset of $\mathcal{E}_4(G)$ of all elements of type \mathbf{C} is denoted by $\mathcal{E}(\mathbf{C})$. We also obtain subsets $\text{Ni}(\mathbf{C}) \subset \text{Ni}_4(G)$ and $\text{Ni}^{\text{red}}(\mathbf{C}) \subset \text{Ni}_4^{\text{red}}(G)$.

In the case $r = 4$, the variety $H_4^{\text{red}}(G)$ is a smooth affine curve, equipped with a finite flat cover

$$j: H_4^{\text{red}}(G) \longrightarrow \mathbb{A}_{\mathbb{Q}}^1,$$

which is ramified at most at $0, 1728$ and étale over $\mathbb{A}_{\mathbb{Q}}^1 - \{0, 1728\}$. The map j is characterized by the following property. Let $t: \text{Spec } k \rightarrow H_4^{\text{red}}(\mathbf{C})$ be a geometric point, corresponding to a G -cover $f: Y \rightarrow X$ with branch locus $D = \{x_1, \dots, x_4\}$. Then $j(t) \in k$ is the j -invariant of the configuration (X, D) .

Let $x_1, \dots, x_4 \in \mathbb{C}$ be four distinct complex numbers. Set $X_0 := \mathbb{P}_{\mathbb{C}}^1$ and $D_0 := \{x_1, \dots, x_4\} \subset X_0$. Let $z_0 \in \mathbb{R}$ denote the j -invariant of the configuration $(\mathbb{P}_{\mathbb{C}}^1, D_0)$. We assume that $z_0 \neq 0, 1728$. After choosing a marking κ of (X_0, D_0) , we obtain a presentation of $\pi_1(U_0)$ with generators $\alpha_1, \dots, \alpha_4$ and relation $\prod_i \alpha_i = 1$. This presentation yields a bijection

$$(18) \quad j^{-1}(z_0) \xrightarrow{\sim} \text{Ni}_4^{\text{red}}(G).$$

(Recall that the left hand side of (18) may be identified with the set of isomorphism classes of G -covers of X_0 with branch locus D_0 .) The fundamental group of $\mathbb{C} - \{0, 1728\}$ acts on the left hand side of (18), and the mapping class group \bar{B}_4 acts on the right hand side. There is a natural identification of these two groups which makes the bijection (18) equivariant. In particular, we obtain a bijection between the set of connected components of $H_4^{\text{red}}(G)_{\mathbb{C}}$ and the \bar{B}_4 -orbits of $\text{Ni}_4^{\text{red}}(G)$.

4.4. We can now use the results of §2.3 to determine the image of η^{top} . We will use the notation introduced in the previous subsection, with the following difference. Since we will be working exclusively with complex analytic spaces, we will omit the index $()_{\mathbb{C}}$. For instance, we write S instead of $S_{\mathbb{C}}$, etc.

Choose a point $s_0 \in S$ with $j(s_0) = z_0$ and let (X_0, D_0) denote the fibre of the configuration (X, D) over s_0 . Let $\rho_0^{\text{top}}: \pi_1^{\text{top}}(U_0, x_0) \rightarrow \text{GL}_n(R)$ denote the restriction of ρ^{top} to the subgroup $\pi_1^{\text{top}}(U_0, x_0) \subset \pi_1^{\text{top}}(U, x_0)$, and set $g_i = \rho_0(\alpha_i) \in G$. By construction, the tuple $\mathbf{g} := (g_i)$ is a generating system for G , and the reduced Nielsen class of \mathbf{g} is an element of the \bar{B}_4 -orbit O . Moreover, $[\mathbf{g}]^{\text{red}}$ is stabilized by the image of the group homomorphism

$$\bar{\varphi}: \pi_1^{\text{top}}(S, s_0) \longrightarrow \bar{B}_4,$$

which is induced by the configuration (X, D) over S .

For simplicity, we also assume that the configuration (X, D) admits an affine frame (Definition 2.4), which we use to identify X with \mathbb{P}_S^1 . This assumption will be satisfied in our main example. Note also that (at least in the situation where S is one-dimensional) there always exists an affine frame over a dense open subset of S .

The ∞ -section defines a section of the natural projection $\pi_1^{\text{top}}(U) \rightarrow \pi_1^{\text{top}}(S)$. We identify $\pi_1^{\text{top}}(S)$ with the image of this section. Let $\chi: \pi_1^{\text{top}}(S) \rightarrow G$ denote the restriction of ρ to $\pi_1^{\text{top}}(S)$. Essentially by definition, we have

$$(19) \quad \mathbf{g}^{\varphi(\gamma)} = \mathbf{g}^{\chi(\gamma)^{-1}},$$

for all $\gamma \in \pi_1(S)$; compare with (17). It follows from Theorem 2.5 that

$$(20) \quad \eta(\gamma) = \bar{\Phi}(\mathbf{g}, \varphi(\gamma)) \cdot \bar{\Psi}(\mathbf{g}, \chi(\gamma)),$$

for all $\gamma \in \pi_1^{\text{top}}(S)$. Here $\bar{\Phi}(\mathbf{g}, \beta)$ and $\bar{\Psi}(\mathbf{g}, h)$ are as in §2.2. Therefore, if we know ϕ and χ explicitly, we can also compute η^{top} .

Remark 4.3: In practice, it is not always so easy to describe an affine frame $X \cong \mathbb{P}_S^1$ and the induced lift φ of $\bar{\varphi}$ explicitly. In many cases, this is possible, using the methods of [6]. However, for applications to the Regular Inverse Galois Problem, it is usually sufficient to determine the image of the projective representation associated to η^{top} , and one can proceed as follows.

Let $\varphi: \pi_1^{\text{top}}(S, s_0) \rightarrow A_4$ and $\chi: \pi_1(S, s_0) \rightarrow G$ be any pair of group homomorphisms such that φ is a lift of $\bar{\varphi}$ and such that (19) holds. (Using the fact that $\pi_1^{\text{top}}(S, s_0)$ is a free group, it is easy to see that such a pair always exists.) The choice of (φ, χ) determines a representation $\rho': \pi_1^{\text{top}}(U) \rightarrow \text{GL}_n(R)$ extending ρ_0 ; it corresponds to a variation \mathcal{V}' of $\mathcal{V}_0^{\text{an}}$. Let $\eta': \pi_1^{\text{top}}(S, s_0) \rightarrow \text{GL}(W_{\mathbf{g}})$ be the representation corresponding to the parabolic cohomology of \mathcal{V}' . By Remark 2.3 (iii), the projective representations associated to η^{top} and η' are equal. See the next section, in particular §5.3.

5. An example

We work out one particular example of the construction described in the last section. In this example, the Hurwitz space is a rational curve. As a result, we obtain regular realizations over $\mathbb{Q}(t)$ of certain simple groups $\text{PSL}_2(\mathbb{F}_{p^2})$.

5.1. Let $G := \text{PSL}_2(7) \times \mathbb{Z}/3\mathbb{Z}$. Given a conjugacy class C of elements of the group $\text{PSL}_2(7)$, we denote by C_i the conjugacy class of (g, i) in G , where $g \in C$ and $i \in \mathbb{Z}/3\mathbb{Z}$. The conjugacy classes of $\text{PSL}_2(7)$ are denoted in the standard way (see [1]). For instance, $2a$ is the unique class of elements of $\text{PSL}_2(7)$ of order 2, and $2a_i$ is the class of (g, i) , with $g \in 2a$. Set

$$\mathbf{C} := (2a_0, 2a_0, 3a_1, 3a_2).$$

A computer calculation shows that the set $\text{Ni}^{\text{red}}(\mathbf{C})$ has 90 elements and that the mapping class group \bar{B}_4 acts transitively. Since \mathbf{C} is *rational* (in the sense of [25]), the connected component $S := H^{\text{red}}(\mathbf{C})$ of the Hurwitz space $H_4^{\text{red}}(G)$ corresponding to this orbit is defined over \mathbb{Q} . So S is a smooth, affine and absolutely irreducible curve over \mathbb{Q} . Furthermore, our explicit knowledge of the braid action on $\text{Ni}^{\text{red}}(\mathbf{C})$ can be used to show that the complete model \bar{S} of S has genus 0

LEMMA 5.1:

- (i) *The curve S is isomorphic to a dense open subset of $\mathbb{P}_{\mathbb{Q}}^1$.*
- (ii) *There exists a versal G -Galois cover $f: Y \rightarrow X$ over S .*

Proof: We have to show that $\bar{S} \cong \mathbb{P}_{\mathbb{Q}}^1$. Since \bar{S} has genus 0, it is well known that it suffices to find a \mathbb{Q} -rational effective divisor of odd degree on \bar{S} . The description of the covering $j: S_{\mathbb{C}} \rightarrow \mathbb{C}$ in terms of the braid action shows that the set of cusps (i.e. the points of $\bar{S} - S$) is such a divisor, of degree 17. This finishes the proof of (i).

Let $s = \text{Spec } k \rightarrow S$ be a geometric point of S and denote by $f_s: Y_s \rightarrow X_s$ the G -cover of type \mathbf{C} corresponding to s . Let x_1, \dots, x_4 denote the branch points of f_s , ordered in such a way that x_i corresponds to the conjugacy class C_i . By definition, we have an injection $G \hookrightarrow \text{Aut}_k(Y_s)$. We claim that the centralizer of G inside $\text{Aut}_k(Y_s)$ is equal to the center of G (which is cyclic of order 3). Indeed, suppose that $\sigma: Y_s \xrightarrow{\sim} Y_s$ is an automorphism which centralizes the action of G . The automorphism $\sigma': X_s \xrightarrow{\sim} X_s$ induced by σ fixes the set $\{x_1, x_2\}$ and the branch points x_3 and x_4 . If σ' were nontrivial, it would be of order 2, and there would exist a reduced Nielsen class $[\mathbf{g}]^{\text{red}} \in \text{Ni}^{\text{red}}(\mathbb{C})$ which is fixed by the element $\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_1 \in \bar{B}_4$. However, one checks that such a Nielsen class does not exist, so σ' is the identity. This proves the claim.

The claim implies that for any G -cover $f: Y \rightarrow X$ over a scheme T whose classifying morphism $\varphi_f: T \rightarrow H_4(G)$ has its image contained in S , the automorphism group of f is canonically isomorphic to the center of G . It is shown in [28] that the category of all (families of) G -covers of type \mathbf{C} is a gerbe over the Hurwitz space $S = H(\mathbf{C})$. In our case, the band of this gerbe is simply the constant group scheme $\mathbb{Z}/3\mathbb{Z}$. By general results on non-abelian cohomology, the gerbe is represented by a class ω in $H^2(S, \mathbb{Z}/3\mathbb{Z})$, and the existence of a global section (i.e. the neutrality of the gerbe) is equivalent to the vanishing of ω . See also [3].

Let K denote the function field of S . By (i), $K = \mathbb{Q}(t)$ is a rational function field. Since S is affine, we may regard ω as an element of the Galois cohomology group $H^2(K, \mathbb{Z}/3\mathbb{Z})$. We can give a more concrete description of ω , as follows. Let $f_{\bar{K}}: Y_{\bar{K}} \rightarrow X_{\bar{K}}$ denote the G -cover of type \mathbf{C} corresponding to the generic geometric point $\text{Spec } \bar{K} \rightarrow S$. For $\sigma \in \text{Gal}(\bar{K}/K)$, let $f_{\bar{K}}^{\sigma}$ denote the conjugate G -cover. By definition of the field K , the cover $f_{\bar{K}}^{\sigma}$ is isomorphic to $f_{\bar{K}}$, i.e.

there exists a commutative diagram

$$\begin{array}{ccc}
 Y_{\bar{K}} & \xrightarrow{\psi_\sigma} & Y_{\bar{K}}^\sigma \\
 f_{\bar{K}} \downarrow & & f_{\bar{K}} \downarrow \\
 X_{\bar{K}} & \xrightarrow{\varphi_\sigma} & X_{\bar{K}}^\sigma,
 \end{array}$$

where ψ_σ and φ_σ are \bar{K} -linear isomorphisms and ψ_σ is also G -equivariant. Note that ψ_σ is not uniquely determined by σ : we may compose it with an element of the center of G . However, φ_σ is uniquely determined by σ and therefore satisfies the obvious cocycle relation. We conclude that there exists a (unique) model X_K of $X_{\bar{K}}$ over K such that φ_σ is determined by the isomorphism $X_{\bar{K}} \cong X_K \otimes \bar{K}$, in the obvious way. In the language of [2], we obtain the following result. The *field of moduli* of the G -cover $f_{\bar{K}}$ with respect to the extension \bar{K}/K and the model X_K of $X_{\bar{K}}$ is equal to K . Moreover, the class $\omega \in H^2(K, \mathbb{Z}/3\mathbb{Z})$ is the obstruction for K to be a field of definition.

The curve X_K is isomorphic to the projective line over K if and only if it has a K -rational point. Moreover, there exists a quadratic extension L/K such that $X_L := X_K \otimes L$ has an L -rational point and is isomorphic to \mathbb{P}_L^1 . It follows from a theorem of Dèbes and Douai [2] that L is a field of definition of $f_{\bar{K}}$ (here we use that the center of G is a direct summand of G). In other words, the restriction of ω to L vanishes. But by [20], Chap. I.2, Prop. 9, the restriction map $H^2(K, \mathbb{Z}/3\mathbb{Z}) \rightarrow H^2(L, \mathbb{Z}/3\mathbb{Z})$ is an isomorphism. We conclude that $\omega = 0$, which finishes the proof of the proposition. ■

Remark 5.2: The lemma shows that there exist infinitely many non-isomorphic G -covers $f_0: Y_0 \rightarrow X_0$ defined over \mathbb{Q} . However, we do not know whether we can find any such G -cover with $X_0 \cong \mathbb{P}_{\mathbb{Q}}^1$. So we do not know whether the lemma produces any regular realizations of the group G over $\mathbb{Q}(t)$.

5.2. Let $\mathbf{g} \in \mathcal{E}(\mathbf{C})$ be any generating system of type \mathbf{C} ; for instance, we could take

$$\mathbf{g} := ((1, 2)(3, 4)(5, 8)(6, 7), (1, 6)(2, 5)(3, 7)(4, 8), \\
 (1, 3, 8)(4, 5, 7)(9, 11, 10), (1, 3, 7)(2, 8, 6)(9, 10, 11))$$

(here we have chosen a faithful permutation representation $G \hookrightarrow S_{11}$). The group G admits a faithful and absolutely irreducible linear representation of dimension 3, defined over the number field $K := \mathbb{Q}(\sqrt{-3}, \sqrt{-7})$. This representation is already defined over $R := \mathcal{O}_K[1/7!]$. From now on, we will consider G

as a subgroup of $GL_3(R)$. Note that the matrices g_1, g_2 are conjugate to the diagonal matrix $\text{diag}(1, 1, -1)$, and that g_3 (resp. g_4) is conjugate to $\text{diag}(1, 1, \omega)$ (resp. $\text{diag}(1, 1, \omega^2)$), with $\omega := (-1 + \sqrt{-3})/2$. Therefore, Remark 1.3 shows that the R -module $W_{\mathbf{g}}$ defined in §1.3 is locally free of rank 2.

We can now apply the construction of §4.2 to the versal G -cover $f: Y \rightarrow X$ of Lemma 5.1. In particular, for each prime ideal \mathfrak{p} of K with residue characteristic $p \geq 11$, we obtain a representation

$$\eta_{\mathfrak{p}}: \pi_1(S, s_0) \longrightarrow GL_2(\mathcal{O}_{\mathfrak{p}}).$$

We let $\eta_{\mathfrak{p}}^{\text{geo}}$ denote the restriction of $\eta_{\mathfrak{p}}$ to the geometric fundamental group $\pi_1(S_{\overline{\mathbb{Q}}})$. We write $\lambda_{\mathfrak{p}}$ and $\lambda_{\mathfrak{p}}^{\text{geo}}$ for the induced projective representations. We say that $\eta_{\mathfrak{p}}$ (resp. $\lambda_{\mathfrak{p}}$) is **regular** if it has the same image as $\eta_{\mathfrak{p}}^{\text{geo}}$ (resp. $\lambda_{\mathfrak{p}}^{\text{geo}}$).

THEOREM 5.3: *Suppose that $p > 7$ is not totally split in the extension K/\mathbb{Q} . Then $\lambda_{\mathfrak{p}}$ is regular and has image $PSL_2(\mathcal{O}_{\mathfrak{p}})$.*

Before we give the proof of this theorem, let us mention the following immediate corollary.

COROLLARY 5.4: *The simple groups $PSL_2(p^2)$ admit regular realizations over $\mathbb{Q}(t)$, for $p \not\equiv 1, 4, 16 \pmod{21}$.*

Note that regular realizations of $PSL_2(p^2)$ are already known for $p \leq 7$. If p is congruent to 1, 4 or 16 modulo 21, then our construction gives a nonregular Galois extension of $\mathbb{Q}(t)$ with group $PGL_2(p)$.

5.3. We now give the proof of Theorem 5.3. Set $\Gamma := \pi_1^{\text{top}}(S, s_0)$; the first step in the proof of Theorem 5.3 is to determine the image of the projective representation $\lambda^{\text{top}}: \Gamma \rightarrow PGL(W_{\mathbf{g}})$ associated to $\eta^{\text{top}}: \Gamma \rightarrow GL(W_{\mathbf{g}})$. Since S is isomorphic to the Riemann sphere minus 17 points, there exist generators $\gamma_1, \dots, \gamma_{17}$ of Γ , subject to the relation $\prod_j \gamma_j = 1$. Our strategy is to explicitly compute the image of γ_j in $PGL(W_{\mathbf{g}})$, for a certain choice of the generators γ_j .

Let $\bar{\varphi}: \Gamma \rightarrow \bar{B}_4$ be the group homomorphism induced from the branch locus configuration (X, D) of the versal G -cover $f: Y \rightarrow X$. By construction, there exists a reduced Nielsen class in $Ni^{\text{red}}(\mathbb{C})$ which is stabilized by the image of $\bar{\varphi}$. Since the action of \bar{B}_4 on $Ni^{\text{red}}(\mathbb{C})$ is transitive, we may normalize things in such a way that the class $[\mathbf{g}]^{\text{red}}$ of our originally chosen tuple \mathbf{g} is stabilized by $\bar{\varphi}(\Gamma)$. It is well known (see e.g. [4]) that there exist generators $\delta_0, \delta_{\infty}, \delta_{1728}$ of $\pi_1^{\text{top}}(\mathbb{C} - \{0, 1728\})$, with relation $\delta_0 \delta_{\infty} \delta_{1728} = 1$, which are mapped to $\bar{\beta}_1 \bar{\beta}_2, \bar{\beta}_1$ and $\bar{\beta}_1 \bar{\beta}_2 \bar{\beta}_1$, under the natural map

$$\pi_1^{\text{top}}(\mathbb{C} - \{0, 1728\}) \longrightarrow \bar{B}_4.$$

Let $\Gamma' \subset \pi_1^{\text{top}}(\mathbb{C} - \{0, 1728\})$ be the inverse image of the stabilizer of the reduced Nielsen class $[\mathbf{g}]^{\text{red}}$. We may identify Γ' with the fundamental group of $S' := j^{-1}(\mathbb{C} - \{0, 1728\}) \subset S$. It is a straightforward, although combinatorially involved problem to write down a list of generators of the free group Γ' , given as words in the generators δ . Moreover, one can choose these generators in such a way that the usual product-1 relation holds and that each of them represents a simple closed loop around one of the points missing from S' . Let $\gamma_1, \dots, \gamma_{17} \in \Gamma'$ be those generators representing a loop around a cusp (i.e. a point $s \in \bar{S}$ with $j(s) = \infty$). Note that γ_j is conjugate (inside the group $\pi_1^{\text{top}}(\mathbb{C} - \{0, 1728\})$) to a certain power of δ_∞ . The other generators, representing a loop around one of the points of $S - S'$, are conjugate either to δ_0^3 or to δ_{1728}^2 , so their image in \bar{B}_4 is 1. It follows that the map $\Gamma' \rightarrow \bar{B}_4$ factors over the natural, surjective map $\Gamma' \rightarrow \Gamma$. Denoting the image of γ_j in Γ by the same name, we have found explicit generators $\gamma_1, \dots, \gamma_{17}$ of Γ , with relation $\prod_j \gamma_j = 1$, and their images under the map $\bar{\varphi}: \Gamma \rightarrow \bar{B}_4$.

It is easy to find, for all $j = 1, \dots, 17$, an element $\gamma'_j \in A_4$ which lifts $\bar{\varphi}(\gamma_j)$ and an element $h_j \in G$ such that

$$\mathbf{g}^{\gamma'_j} = \mathbf{g}^{h_j}.$$

Moreover, we may do this in such a way that $\prod_j \gamma'_j = 1$ and $\prod_j h_j = 1$. In other words, we can choose homomorphisms $\varphi: \Gamma \rightarrow A_4$ and $\chi: \Gamma \rightarrow G$ as in Remark 4.3. In fact, the lift φ is unique, because the Klein four group Q acts faithfully on $\text{Ni}(\mathbb{C})$. On the other hand, χ is only determined up to multiplication of $h_j = \chi(\gamma_j)$ by a central element of order 3. This corresponds to the fact that the versal G -cover $f: Y \rightarrow X$ over S may be twisted by characters of order 3. (It is not clear how to find χ corresponding to a versal cover f defined over \mathbb{Q} .) By formula (20) and Remark 4.3 we have

$$\eta^{\text{top}}(\gamma_j) = c_j \cdot \bar{\Phi}(\mathbf{g}, \gamma'_j) \cdot \bar{\Psi}(\mathbf{g}, h_j),$$

for some scalar $c_j \in K^\times$. (In fact, c_j is a third root of unity and we have $\prod_j c_j = 1$.) Set $b_j := \bar{\Phi}(\mathbf{g}, \gamma'_j) \cdot \bar{\Psi}(\mathbf{g}, h_j)$. By construction, b_j is an invertible 2-by-2 matrix with entries in R such that $\prod_j b_j = 1$.

Using a computer program written in *GAP*, the authors have computed the matrices b_j explicitly. It turns out that 12 of the b_j are transvections and 5 are homologies with eigenvalues $1, \omega$ or $1, \omega^2$, where ω denotes a primitive third root of unity (see [10] for notation). One finds that the trace of the matrix $b_1 b_2$ is a generator of the extension K/\mathbb{Q} . Moreover, one checks that for every

prime $p > 7$ one can find a pair of transvections b_i, b_j whose commutator is not congruent to the identity, modulo any prime ideal \mathfrak{p} above p . This information suffices to show that for a prime $p > 7$ which is not totally split in K/\mathbb{Q} , the image of the residual projectivized representation $\bar{\lambda}_{\mathfrak{p}}^{\text{geo}}$ associated to $\eta_{\mathfrak{p}}^{\text{geo}}$ is equal to $\text{PSL}_2(p^2)$ (we may identify the residue field of \mathfrak{p} with \mathbb{F}_{p^2}). By a well-known argument (see e.g. [21]), it follows that the image of the projective representation $\lambda_{\mathfrak{p}}^{\text{geo}}$ associated to $\eta_{\mathfrak{p}}^{\text{geo}}$ is equal to $\text{PSL}_2(\mathcal{O}_{\mathfrak{p}})$.

The only thing left to prove is that the image of the full projective representation $\lambda_{\mathfrak{p}}$ is equal to $\text{PSL}_2(\mathcal{O}_{\mathfrak{p}})$ as well. Again, it suffices to show that the image of the residual projectivized representation $\bar{\lambda}_{\mathfrak{p}}$ is equal to $\text{PSL}_2(p^2)$.

One observes that there are exactly five ramification points $s_1, \dots, s_5 \in \bar{S}$ of the map $j: \bar{S} \rightarrow \mathbb{P}^1$ above ∞ whose ramification index is equal to 4. One also observes that the matrices $b_{j_{\mu}}$ corresponding to the points s_{μ} are transvections for $\mu = 1, \dots, 4$, whereas b_{j_5} is a homology. It follows that the set $\{s_1, \dots, s_4\}$ is rational, i.e. fixed by the action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Furthermore, the transvections $b_{\mu_1}, \dots, b_{\mu_4}$ are all conjugate to each other by elements of $\text{SL}_2(K)$. One concludes that the image of these transvections gives rise to conjugate transvections in the image of $\bar{\eta}_{\mathfrak{p}}^{\text{geo}}$. The conjugacy class of these transvections is a rational class, in the sense of [25].

Suppose that there exists an element $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and a lift $\alpha \in \pi_1(S)$ of σ such that $\bar{\eta}_{\mathfrak{p}}(\alpha) \in \text{GL}_2(p^2)$ does not lie in $\text{SL}_2(p^2)Z(\text{GL}_2(p^2))$. Using the branch cycle argument (as in the proof of [27], Corollary 4.6), one would conclude that σ does not fix the set $\{s_1, \dots, s_4\}$. But this would be a contradiction to the assertion made above. It follows that the image of $\bar{\lambda}_{\mathfrak{p}}$ is equal to $\text{PSL}_2(p^2)$. The proof of Theorem 5.3 is now complete. ■

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